8(c) Obtain the eigen values and eigen vectors of the matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ and verify that the eigen vectors are orthogonal.

Soln- Characteristic eqn is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)^2] + 1[-1(2-\lambda)] = 0$$

$$(2-\lambda)^3 - 2 + \lambda = 0$$

$$-\lambda^3 + 6\lambda^2 - 12\lambda + 8 - 2 + \lambda = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 1, 3, 2$$

Eigen values are 1, 2, 3.

Eigen vector corresponding to $\lambda = 1$

$$[A-I][X_1] = 0$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 + 0x_2 + x_3 = 0$$

$$0x_1 + x_2 + 0x_3 = 0$$

$$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

Eigen vector corresponding to $\lambda = 2$.

$$[A-2I][X_2] = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}$$

$$x_1 + 0x_2 + x_3 = 0$$
$$0x_1 + x_2 + 0x_3 = 0$$
$$\frac{x_1}{-1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

Eigen vector corresponding to $\lambda = 2$.

$$[A - 2I][X_2] = 0$$
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$0x_1 + 0x_2 + x_3 = 0$$
$$x_1 + 0x_2 + 0x_3 = 0$$
$$\frac{x_1}{0} = \frac{-x_2}{-1} = \frac{x_3}{0}$$

Eigen vector corresponding to $\lambda = 3$

$$[A - 3I][X_3] = 0$$
$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
$$-x_1 + 0x_2 + x_3 = 0$$
$$0x_1 - x_2 + 0x_3 = 0$$
$$\frac{x_1}{1} = \frac{-x_2}{0} = \frac{x_3}{1}$$

Let $P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad P^T = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$P^T P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$