

characteristic eqⁿ is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)(2-\lambda)] - 1[0] + 1[-1+\lambda] = 0$$

$$(2-\lambda)[2-3\lambda+\lambda^2] - 1 + \lambda = 0$$

$$4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

$$A^3 = \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} \quad A^2 = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$= A^3 - 5A^2 + 7A - 3I$$

$$= \begin{bmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{bmatrix} - \begin{bmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, theorem verified.

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^2 - 5A + 7I = 3A^{-1}$$

$$\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 3A^{-1}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} &= A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ &= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2) + 8A^2 - 2A + I \\ &= A(3I - 7A) + 8A^2 - 2A + I \\ &= 3A - 7A^2 + 8A^2 - 2A + I \\ &= A^2 + A + I \end{aligned}$$